Chirality in unified theories of gravity

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Abstract. We show how to obtain a single chiral family of an $SO(10)$ GUT, starting from a Majorana-Weyl representation of a unifying ("GraviGUT") group $SO(3,11)$, which contains the gravitational Lorentz group $SO(3,1)$. An action is proposed, which reduces to the correct fermionic GUT action in the broken phase.

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INTRODUCTION

Low energy chirality poses strong constraints on unified model building. For example, in Grand Unified Theories (GUT) the fermionic multiplet must be in a complex representation of the gauge group. At the same time, chirality precludes the use of orthogonal groups larger than $SO(10)$ or exceptional groups larger than E_6 [1]. This is relevant for instance when one tries to put all fermionic families in a single spinor multiplet [2, 3]. The interplay between Lorentz and internal representations becomes trickier when gravity is involved. In the Kaluza–Klein approach to unification, it is difficult to obtain chiral fermions in four dimensions, even starting from chiral representations in higher dimensions [4]. In string theory chirality of the low energy degrees of freedom is achieved by suitably choosing the topology of the compact dimensions, but then unification comes at the cost of introducing infinitely many new local degrees of freedom. More recently, an ambitious attempt to unify all known fields into a single representation of E_8 [5] stumbled into chirality issues [6].

Here we discuss the issue of chirality in the context of theories where the Lorentz group, which is gauged in theories of gravity, is unified with a GUT group in a larger group G. By this we mean that the gravitational connection and the gauge fields of a GUT are components of a connection for the unifying group G. We will call such a theory a "GraviGUT" (GGUT). Unlike in [5], we do not insist on putting all fields in a single representation of G: gravitons, gauge fields, fermions and scalars will belong to different multiplets. The general idea for this kind of unification has been discussed in [7–9]. It is a rather natural generalization of the GUT program, encompassing also gravitational interactions. The main difference is that the order parameter cannot be a scalar but must include a multiplet of one forms, called the soldering form.[1] In [7] the use of $G = SO(1,13)$ was proposed, where the soldering form $\theta^i{}_\mu$ with $i = 1,\dots,14$ and $\mu = 1,2,3,4$ is in the fundamental representation.[2] If the dynamics generates a VEV for θ which has rank 4, then one can choose a gauge where $\theta^i{}_\mu = 0$ for $i = 5,\dots,14$. This "unitary gauge" breaks the original gauge group to $SO(10)$, and the breaking scale is identified with the Planck scale. The Lorentz and mixed parts of the connection all become massive at this scale, explaining why we do not see these degrees of freedom at low energies.

As a preliminary step, in [8] we discussed mainly the possibilities for a unification of gravity with the weak interactions. This 'graviweak' unification is also the basis for a model of geometrical origin [9] that predicts also the right strong interactions, but at the price of duplicating the unified gravitational sector at high energy. Here we want to include also the strong interactions in a single unified group. Probably the most promising path towards this unification is via the Pati-Salam model [10], based on the group $SU(2)_L \times SU(2)_R \times SU(4)$. In view of the fact that this group is locally isomorphic to $SO(4) \times SO(6)$, and that the Lorentz group is also (pseudo)-orthogonal, it seems natural to choose G to be a pseudo-orthogonal group $SO(p,q)$ with $p+q = 14$. In order to accommodate the Pati-Salam and Lorentz groups, the possibilities are restricted to $SO(1,13)$, $SO(3,11)$, $SO(5,9)$, $SO(7,7)$. In the latter two cases the weak and strong gauge fields would belong to subalgebras with different signature, so that a standard Yang-Mills action would lead to ghosts. We will restrict our attention to the remaining two possibilities, which thus contain the full $SO(10)$ GUT.

It has already been noticed [7], for the case $G = SO(1,13)$, that the fermion multiplets occurring at low energy lend support to this unification scenario. In fact the **64**, chiral spinor representation of $SO(1,13)$, breaks

[1] In some formulations inspired by the Plebanski formalism it may be preferable to use a two form, dynamically equivalent to the soldering form on shell [12].

[2] We observe here that the soldering form used in the graviweak unifications [8, 9] carried the tensor product of two vector representations, because the fermions were in the vector representation of the group. In [7] as well as in the present work the fermions are in a spinor representation therefore their tensor product contains the fundamental as an invariant subspace, and it is consistent to restrict oneself to it.

under the subgroup $SO(1,3) \times SO(10)$ into $(\mathbf{2}, \mathbf{16}) \oplus (\overline{\mathbf{2}}, \overline{\mathbf{16}})$. The fact that the known fermions are spinors of Lorentz and spinors of $SO(10)$ would thus be naturally explained. Here we will consider in greater detail the case $G = SO(3,11)$, which admits Majorana-Weyl spinors. In section II we will show by explicit construction that one such representation gives rise to a single standard model family, which can be identified with a chiral $(\mathbf{2}, \mathbf{16})$. In so doing we find the transformations that relates a basis for the Clifford algebra of $SO(3,11)$ to a basis which is adapted to the subgroup $SO(3,1) \times SO(10)$. This will allow us, in the section III, to write the kinetic term for the fermions in an $SO(3,11)$–invariant way, and to see how it reduces to the familiar one at low energy. We also show that the mixed (Lorentz-GUT) gauge fields mediate new high energy processes. In section IV we conclude with some further comments.

$SO(3,11)$ SPINORS AND GGUT

We start from a set of (128-dimensional, complex) gamma matrices γ_i for $SO(3,11)$ given explicitly in the Appendix, and the corresponding chirality operator $\hat{\gamma} = \Pi_{i=1}^{14} \gamma_i$ and algebra generators $\Sigma_{ij} = \frac{1}{4}[\gamma_i, \gamma_j]$. It is a property of the Dirac representation that it is equivalent to its hermitian conjugate, its complex conjugate and its transpose. These equivalences are realized by three intertwining operators A, B, C, defined by:

$$\Sigma_{ij}^\dagger A = -A\,\Sigma_{ij}\,, \quad \Sigma_{ij}^t C = -C\,\Sigma_{ij}\,, \quad B\,\Sigma_{ij}^* = \Sigma_{ij}\,B\,.$$

The matrices A and C can be used to construct the invariant hermitian and bilinear forms $\psi_1^\dagger A \psi_2$ and $\psi_1^t C \psi_2$. The matrix B defines charge conjugation $\psi^c = (B \circ \star)\psi \equiv B\psi^*$, which for $SO(3,11)$ is an antilinear involution because $BB^* = \mathbf{1}$. One can thus define the left/right eigenspaces of $\hat{\gamma}$ by $\hat{\gamma}\psi_{L/R} = \mp \psi_{L/R}$ and the $+/-$ eigenspaces of charge conjugation by $(\psi_\pm)^c = \pm \psi_\pm$. An important property of $SO(3,11)$ is that the matrices $\hat{\gamma}$ and B commute. Thus one can define simultaneous eigenspaces of chirality and charge conjugation, i.e. Majorana-Weyl (MW) spinors

It is possible and convenient to choose a basis that is adapted to the MW representation, in the sense that

$$A = C = \mathbf{1}_{64} \otimes \sigma_1\,, \quad B = \mathbf{1}_{128}\,, \quad \hat{\gamma} = -\mathbf{1}_{64} \otimes \sigma_3\,. \quad (1)$$

In this basis charge conjugation is just complex conjugation, and the MW spinors are just the real and imaginary parts of chiral spinors: $\psi_L = \psi_{L+} + i\psi_{L-}$ (and similarly for R). It is then useful to define a map $\mathcal{R} : \mathbb{C}^n \to \mathbb{R}^{2n}$ from complex n-vectors to real $2n$ vectors by $\mathcal{R}v = (\mathrm{Re}\, v, \mathrm{Im}\, v)^t$, and the inverse map which associates to the vector $w = (w_1, w_2)^t \in \mathbb{R}^{2n}$ the vector $\mathcal{R}^{-1}w = w_1 + iw_2$. Using these maps, we can view the MW spaces either as complex 32-dimensional or real 64-dimensional representations.

We wish to identify a standard model fermion family with a single MW representation of $SO(3,11)$, for example with the 64 real degrees of freedom of ψ_{L+}. Then, we need to show that, decomposed as representations of $SO(3,1) \times SO(10)$, these describe precisely the 32 complex components of a chiral spinor of Lorentz and chiral spinor of $SO(10)$, i.e. the representation $(\mathbf{2}, \mathbf{16})$.

In order to do this, one has to pick half of the components of ψ_{L+} and use them as real parts of a complex $SO(10)$ spinor, while the remaining components give the imaginary parts. There is no natural way of doing this; in fact, any such operation corresponds to a choice of a complex structure in $\mathbb{R}^{64}$. The simplest choice would be $\mathcal{R}^{-1}\psi_{L+}$, but one should not expect it to have simple transformation properties under the subgroup $SO(3,1) \times SO(10)$. However, there exist a real (64×64) orthogonal transformation W_L such that $\mathcal{R}^{-1} W_L \psi_{L+}$ do. To find it, we impose that 51 of the $SO(3,11)$ generators match those of $SO(3,1) \times SO(10)$ in the respective (left) Weyl bases:

$$\mathcal{R}^{-1} W_L \Sigma^{(3,11)}_{L\,ij} W_L^t \mathcal{R} = \begin{cases} \Sigma^{(3,1)}_{L\,mn} \otimes \mathbf{1}_{16} & \text{for } ij = mn \\ \mathbf{1}_2 \otimes \Sigma^{(10)}_{L\,ab} & \text{for } ij = ab\,. \end{cases} \quad (2)$$

(We use indices $m, n = 1, 2, 3, 4$ and $a, b = 5, \ldots 14$.) We find that the matrix W_L is almost completely determined by these equations, up to a free angle α. Note that we do not impose any requirement on the remaining 40 generators, $\Sigma^{(3,11)}_{L\,ma}$, mixing Lorentz and $SO(10)$ subspaces.

We have thus found the explicit transformation between a single MW spinor ψ_{L+} of $SO(3,11)$ and a Weyl spinor $\eta_{(\mathbf{2},\mathbf{16})}$ of $SO(3,1) \times SO(10)$, representing a family in a $SO(10)$ GUT:

$$\eta_{(\mathbf{2},\mathbf{16})} = \mathcal{R}^{-1} W_L\, \psi_{L+}\,. \quad (3)$$

It is useful to observe that the operator W_L is not linear with respect to the chosen complex structure. Inverting the above relation, an antilinear part emerges:

$$\psi_{L+} = W_L^t \mathcal{R}\, \eta_{(2,16)} = \mathcal{R}(X_W \eta_{(2,16)} + Y_W \eta^*_{(2,16)})\,. \quad (4)$$

where X_W and Y_W are certain complex matrices. A consequence of this is that not all generators of $SO(3,11)$ can be realized linearly on the spinors $\eta_{(2,16)}$: by construction the Lorentz and $SO(10)$ generators act linearly (they are a representation!) but the generators that mix Lorentz and $SO(10)$ turn out to be antilinear. For later reference, they can be written as

$$\mathcal{R}^{-1} W_L \Sigma^{(3,11)}_{L\,ma} W_L^t \mathcal{R} = \frac{\mathrm{e}^{2i\alpha}}{2} (CA\gamma_m)_L^{(3,1)} \otimes (C\gamma_a)_L^{(10)} \circ \star\,. \quad (5)$$

We have obtained a $(\mathbf{2}, \mathbf{16})$ family of fermions starting from the MW representation ψ_{L+} of $SO(3,11)$. In order to understand the fate of the other MW representations,

we need two more facts. The first is that, when (2) holds for ψ_{L+}, for ψ_{R+} we have

$$\mathcal{R}^{-1}W_R\Sigma_{R\,ij}W_R^t\mathcal{R} = \begin{cases} \Sigma_{L\,mn}\otimes \mathbf{1}_{16} & \text{for } ij=mn \\ \mathbf{1}_2\otimes\Sigma_{R\,ab} & \text{for } ij=ab\,, \end{cases} \qquad (6)$$

with $W_R = W_L$. Therefore ψ_{R+} can be identified with $(\mathbf{2},\overline{\mathbf{16}})$. Next we introduce the parity operation, in such a way that in the broken phase it reduces to spatial parity, i.e. a matrix that anticommutes with the three space-like γ's. In our MW basis it is:

$$P_{(3,11)} = i\gamma_1\gamma_2\gamma_3\hat{\gamma} = \mathbf{1}_{64}\otimes\sigma_2\,, \qquad (7)$$

where the phase has been chosen so that $P^2_{(3,11)} = \mathbf{1}$. Since it is imaginary, we see that it exchanges not only the Weyl subspaces, but also the Majorana sectors: $P\psi_{L\pm} = \pm\psi_{R\mp}$. Since spatial parity maps $(\mathbf{2},\mathbf{16})$ to $(\overline{\mathbf{2}},\mathbf{16})$, we have the identification of $(\psi_{L+},\psi_{L-},\psi_{R+},\psi_{R-})$ with $(\eta_{(\mathbf{2},\mathbf{16})},\eta_{(\overline{\mathbf{2}},\overline{\mathbf{16}})},\eta_{(\mathbf{2},\overline{\mathbf{16}})},\eta_{(\overline{\mathbf{2}},\mathbf{16})})$.

As a check, the action of $P_{(3,11)}$ on the subspace of the $SO(3,1)\times SO(10)$ Dirac spinors $\eta = (\eta_{(\mathbf{2},\mathbf{16})},\eta_{(\overline{\mathbf{2}},\mathbf{16})})$ is found to be simply the spacetime parity γ_4:

$$P = \mathcal{R}^{-1}W\,P_{(3,11)}\,W^t\mathcal{R} = \mathbf{1}_{32}\otimes\sigma_2 = \mathbf{1}_{16}\otimes\gamma_4\,. \qquad (8)$$

Thus, in the broken phase parity is inherited by the Lorentz group.

Let us pause to discuss the physical meaning of these group theoretic results. It is instructive to think of them from an $SO(10)$ GUT perspective. Each family of fermions is a $(\mathbf{2},\mathbf{16})$ complex, chiral representation of $SO(3,1)\times SO(10)$, where $SO(3,1)$ is the Lorentz group. We have shown that the fields in such a representation can be rearranged into a real vector and when this is done they are seen to carry not only a representation of $SO(3,1)\times SO(10)$, but of the larger group $SO(3,11)$. We have therefore successfully identified a group that can be used to unify the gravitational and GUT gauge sectors. The reason why the existence of this group is not evident in the original complex form is that the generators that are not in $SO(3,1)\times SO(10)$ act antilinearly on the fields. All the generators form nevertheless a perfectly well defined real representation, namely the MW $\mathbf{64}$ of $SO(3,11)$.

This construction evades the restrictions that chirality of the low energy spectrum poses on extensions of GUT theories, which were mentioned in the introduction. First, it is clear that chiral fermions that are in a real (or pseudoreal) representations of a GUT group would always lead to a nonchiral theory, therefore fermions must be in complex representations. Then one has to avoid the appearance of antifamilies, which would also be in disagreement with the chirality of the spectrum of the standard model. It is in fact not possible to make antifamilies unobservable by giving them a very large mass ($\gg$ TeV) because any mechanism giving mass to a chiral (anti)family at some high energy scale would necessarily break at least the weak $SU(2)$ symmetry at that scale. Therefore also antifamiles should have mass near the electroweak scale, where there are quite strong constraints on their observation.[3]

We also recall that the problem of antifamilies always arises for orthogonal GUT groups larger than $SO(10)$. For instance, the MW representation $\mathbf{128}$ of the group $SO(16)$ has been used in an attempt at family unification [3]. Under the breaking $SO(16)\to SO(6)\times SO(10)$ it decomposes as $\mathbf{128}\to(\mathbf{4},\mathbf{16})\oplus(\overline{\mathbf{4}},\overline{\mathbf{16}})$, where $\mathbf{4}$ is the chiral spinor of $SO(6)$. The second factor represents four $\overline{\mathbf{16}}$ multiplets of the same Lorentz chirality of the $\mathbf{16}$, i.e. four antifamilies, showing that the theory is nonchiral. A further problem in this model is that for $SO(16)$ a mass term of the form $\psi^t_{L+}C_{(3,1)}C_{(16)}\psi_{L+}$ is allowed, because the matrix $C_{(16)}$ is block-*diagonal* in Dirac space. Thus one needs additional symmetries to protect the spinors from a large (Planck or GUT-size) mass term.

For the MW spinors of the $SO(3,11)$ GGUT suggested here, these problems are both absent. First, because Lorentz is included in the unification, the real representation of the GGUT group is actually a single complex representation of the GUT group. Second, any bare mass term is forbidden. This can be seen directly as a consequence of chirality of $SO(3,11)$: because the matrix $C_{(3,11)}$ is block-*anti*diagonal in Dirac space, then $\psi^t_{L+}C_{(3,11)}\psi_{L+}=0$.[4]

Thus, we have shown that GGUTs can be chiral by construction, in spite of adopting real representations and orthogonal groups larger than $SO(10)$. In particular, chirality of the GGUT representation is maintained at low energy. By using $SO(3,11)$ and its Majorana-Weyl representation one can achieve a single standard model family, while by using the chiral representation of $SO(1,13)$ one could generate *two* standard model families. It is also clear that if we started from a nonchiral (Dirac) representation of $SO(3,11)$ we would have ended with two families and two antifamilies.

DYNAMICS

Constructing an action for a GGUTs poses new challenges that go beyond those familiar in GUTs. One would like to have an action which is well defined both in the symmetric and broken phase of the theory. But in these theories the symmetric phase is topological (the metric $\theta^i{}_\mu\theta^j{}_\nu\eta_{ij}$ vanishes classically) so one cannot use the standard type of actions. In [7] a sort of mean field dynamics

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[3] Of course, if antifamilies were discovered in the future below the TeV scale (e.g. [15]) this restriction would have to be reviewed.

[4] Also vanishing because $C_{(3,11)}$ is symmetric while ψ are anticommuting.

was proposed, generating the VEV of θ selfconsistently. Another approach is to use techniques which have been studied in the context of topological theories. We concentrate here only on the action for the fermions.

We begin by defining the $SO(3,11)$ covariant derivative acting on MW spinors

$$D_\mu \psi_{L+} = \left(\partial_\mu + \frac{1}{2} A_\mu^{ij} \Sigma^{(3,11)}_{L+\,ij}\right) \psi_{L+}\,. \tag{9}$$

Note that $\Sigma^{(3,11)}_{L\pm\,ij} = \Sigma^{(3,11)}_{L\,ij}$ are real. Then we define the covariant differential D, mapping spinors to spinor-valued one forms: $D\psi_{L+} = D_\mu \psi_{L+} dx^\mu$. The quadratic form

$$\psi^\dagger_{L+} (A\gamma^i)_L D\psi_{L+} \tag{10}$$

is manifestly a vector under $SO(3,11)$ and a one form under diffeomorphisms.[5] Then, to construct a $SO(3,11)$-invariant action, we introduce an auxiliary field $\phi_{ijk\ell}$ transforming as a totally antisymmetric tensor. The action is

$$\mathcal{S} = \int \psi^\dagger_{L+} (A\gamma^i)_L D\psi_{L+} \wedge \theta^j \wedge \theta^k \wedge \theta^\ell \, \phi_{ijk\ell}\,. \tag{11}$$

The breaking of the $SO(3,11)$ group to the Lorentz and $SO(10)$ subgroups is induced by the VEV of two fields: the soldering one-form θ^i_μ and the four-index antisymmetric field $\phi_{ijk\ell}$.[6] We assume that the VEV of $\phi_{ijk\ell}$ is ϵ_{mnrs}, the standard four-index antisymmetric symbol, in the Lorentz subspace, and zero otherwise. The VEV of the soldering form on the other hand has maximal rank (four) and is also nonvanishing only in the Lorentz subspace, $m = 1,2,3,4$:

$$\begin{cases} \phi_{mnrs} = \epsilon_{mnrs} \\ \phi_{ijk\ell} = 0 \quad \text{otherwise} \end{cases} \qquad \begin{cases} \theta^m_\mu = M e^m{}_\mu \\ \theta^a_\mu = 0 \quad \text{otherwise} \end{cases} \tag{12}$$

where $e^m{}_\mu$ is a vierbein, corresponding to some solution of the gravitational field equations which we need not specify in this discussion (below we will choose $e^m{}_\mu = \delta^m_\mu$) and M can be identified with the Planck mass. Clearly the breaking pattern just described is the one that leads to a theory which is Lorentz invariant (at each point) but other choices may be possible (see comments below).

Using (4) and omitting the subscript $(\mathbf{2},\mathbf{16})$ from the spinors, the kinetic quadratic form (10) becomes

$$\eta^\dagger \mathcal{R}^{-1} W_L (A\gamma^i)_L D W^t_L \mathcal{R}\, \eta\,. \tag{13}$$

[5] The product $A\gamma^i$ is block diagonal in Dirac space, because both A and γ_i are block *anti*-diagonal.

[6] The field ϕ_{ijkl} also appears in Plebanski reformulations of General Relativity, where the vierbein field is traded for a two form field. If the (Lorentz) gauge group is extended, ϕ serves, as in the present context, to achieve the symmetry breaking [12].

In the broken phase, treating separately the cases $i = m = 1,2,3,4$ and $i = a = 5,\ldots 14$, we find:

$$\mathcal{R}^{-1} W_L (A\gamma^m)_L W^t_L \mathcal{R} = i (A\gamma^m)^{(3,1)}_L \otimes \mathbf{1}_{16} \tag{14}$$

$$\mathcal{R}^{-1} W_L (A\gamma^a)_L W^t_L \mathcal{R} = i e^{2i\alpha} C^{(3,1)}_L \otimes (C\gamma^a)^{(10)}_L \circ \star\,. \tag{15}$$

Therefore, using (14) and the fact that for Lorentz $(A\gamma^m)^{(3,1)}_L = \sigma^m$, together with (2) for the connection terms in the covariant derivative, the action with a flat background vierbein reduces to the standard one for a $SO(10)$ family in flat space:

$$\int d^4x\, \eta^\dagger \sigma^\mu \nabla_\mu \eta\,, \tag{16}$$

where now $\nabla_\mu = D^{(10)}_\mu = \partial_\mu + \frac{1}{2} A^{ab}_{\mu\,(10)} \Sigma^{(10)}_{ab}$ is the $SO(10)$ covariant derivative. Note that this action contains the standard kinetic term of the fermions, and the interaction with the $SO(10)$ gauge fields, which at this stage can still be assumed to be massless.

Had we chosen a nonflat gravitational background, the action would contain the invariant volume factor $|e|$ and the covariant derivative would also contain a nontrivial Lorentz part: $\nabla_\mu = D^{(10)}_\mu + \frac{1}{2} A^{mn}_{\mu\,(3,1)} \Sigma^{(3,1)}_{mn}$. As discussed in [7], the Lorentz connection $A^{mn}_{\mu\,(3,1)}$ in the covariant derivative can be assumed to be the Levi-Civita connection derived from the vierbein. Its fluctuations around this VEV are also present but have a mass of the order of the Planck mass and are negligible at low energies.

The remaining A^{ma}_μ components of the $SO(3,11)$ connection, that mix Lorentz and $SO(10)$, also have Planck mass. These gauge fields, carrying a Lorentz and a $SO(10)$ vector index, can be decomposed in Lorentz representations by lowering the m index with a vierbein, leading to the two fields $A^a_{(\mu\nu)}$, $A^a_{[\mu\nu]}$. They contain thus a symmetric and an antisymmetric field, both in the representation $\mathbf{10}$ of $SO(10)$, that interact with fermions via the following vertex:

$$\begin{aligned} & e^{2i\alpha} A^{ma}_\mu \, \eta^t [(C\gamma^\mu \gamma_m)^{(3,1)}_L \otimes (C\gamma_a)^{(10)}_L] \eta = \\ & \quad = e^{2i\alpha}\, \eta^t [C^{(3,1)} (A^a_{(\mu\nu)} g^{\mu\nu} + A^a_{[\mu\nu]} \sigma^{\mu\nu}) \otimes (C\gamma_a)^{(10)}_L] \eta\,. \end{aligned} \tag{17}$$

The first of the two vertices is equivalent to the one generated by the standard scalar Higgs field $\mathbf{10}$ of $SO(10)$, while the second is a new vertex that involves the spin. The resulting four fermion interactions may lead to new gravitational contributions to rare processes.

We observe that even though these new interactions originate from the generators mixing $SO(10)$ and Lorentz indices, if the breaking works as above, global Lorentz symmetry is not broken by these interactions, because the original lagrangian has local Lorentz symmetry as a (subgroup of the) gauge symmetry, and the background VEVs (12) preserve the global spacetime remnant of this gauge symmetry (see [8, 17] for a detailed discussion).

SUMMARY AND OUTLOOK

A GraviGUT is a very natural extension of a GUT, encompassing also gravitational interactions. Given that the (pseudo)orthogonal group plays a fundamental role in the theory of gravity, it is especially attractive to consider GGUTs that are (pseudo)-orthogonal extensions of an $SO(10)$ GUT. The minimal theory of this type can be based on $SO(1,13)$ or $SO(3,11)$. We have shown that the latter choice is slightly more natural from the point of view of the fermionic content, because it can accommodate three families, whereas $SO(1,13)$ leads to an even number of families. The field content of the simplest GGUT would thus be an $SO(3,11)$ Yang-Mills field, three Majorana-Weyl fermions plus whatever is needed to break the original symmetry to what we see at low energy. The first step of the symmetry breaking chain is essentially unique: $SO(3,11) \to SO(3,1)\times SO(10)$. This is achieved by postulating a nontrivial VEV for a suitable order parameter. The distinctive feature of this first step is that the order parameter is not a scalar but rather a one form with values in the vector representation of the gauge group, θ^i_μ. This so called soldering form provides the necessary connection between spacetime and internal transformations, and its first four components θ^m_μ carry the gravitational degrees of freedom in the broken phase.

At this stage it is less clear what degrees of freedom are needed to describe the further breaking of $SO(10)$ to the standard model group $SU(3)\times SU(2)\times U(1)$, and the final breaking of the latter to the electromagnetic $U(1)$. This will have to be investigated in the future. In principle requiring the GGUT representations to decompose into well-behaved states at low energy, together with the restrictive choice of a GGUT group, should pose constraints also on the GUT sector. At the same time we observe that the breaking of the GUT group is anyway an open issue (see e.g. [20] for a recent thorough reanalysis of non-SUSY $SO(10)$), and that even in the context of the Standard Model the origin of the electroweak symmetry breaking is still partly shrouded in mystery. So it should not come as too much of a surprise if this sector of the GGUT is also less understood.

In the present paper we have discussed in detail the kinematics (sect. II) and dynamics (sec. III) of the fermionic sector. In particular, in section II, we have shown explicitly the equivalence beween the MW representation **64** of $SO(3,11)$ and the $(\mathbf{2},\mathbf{16})$ chiral, complex spinor representation of Lorentz and $SO(10)$, representing a family of Standard Model fermions. This identification evades the problems that chirality of the Standard Model spectrum poses to unified theories, and thus $SO(3,11)$ can be safely adopted as a basis for a unified theory. A further consequence of this construction is that $SO(3,11)$ is also the largest (pseudo-orthogonal) group allowing a chiral low energy spectrum, and thus attempts to achieve family unification by further enlargement of the group are not possible in this approach without introducing mirror families. In section III we have then constructed a diffeomorphism- and $SO(3,11)$-invariant action for fermions and shown how, under a suitable symmetry breaking realized by means of the soldering form and an additional antisymmetric tensor field, this reduces to the correct $SO(10)$-invariant action coupled to gravity at low energy. Various hurdles will have to be overcome in the development of GGUTs, but we have shown here that the construction of a realistic fermionic sector is not an obstacle. We can thus claim that, at least on this count, the setup described here represents the first realistic framework that unifies gravity with the other known interactions. In the rest of this section we discuss a few of the open issues.

The bosonic part of the action, including the gauge and Higgs terms, is probably the most important omission. In a less ambitious form of unification, it has been discussed in [8, 9], see also [12–14] and, for a completely different approach, [7]. In this connection, an issue that is sometimes raised is the presence of ghosts: given that the gauge group is noncompact, one expects that some components of the connection will have wrong sign kinetic terms. Surely, one wants to avoid ghosts in the low-energy GUT gauge sector: this problem was already mentioned in the introduction, and we used it to select some group rather than others. The GGUT groups we discarded would have led to ghosts with a mass of the order of the GUT or lower, while the groups we selected would seem naively to have ghosts with Planck mass. This is what happens also in generic gravitational theories with propagating torsion, independent of unification [16]. Over time, there have been various proposals to circumvent this problem [18, 19]. Here we may add that since the ghosts would occur near or beyond the transition to a different, topological phase, the standard tree level analysis is certainly not conclusive.

The detailed phenomenology of a GGUT will depend upon the details of the symmetry breaking chain. As in ordinary GUTs, the most characteristic signal will come from new interactions mediated by the components of the gauge field on the broken generators of the GGUT group, in the present case the heavy gauge fields mixing Lorentz and $SO(10)$ indices. Their effect is similar to that of a $SO(10)$ higgs field in the representation **10**. We have shown that the corresponding generators are antilinear, and these processes will violate fermion number by two units. One can expect that interactions similar to proton decay and neutron-antineutron oscillations would be present, but with new spin structure. These interactions would be suppressed by the large mass of A^{ma}_μ, so only extremely rare processes would have a chance of being observable.

The symmetry breaking VEVs that we proposed conserve the Lorentz symmetry, but it is conceivable that, with different VEVs of θ and ϕ, Lorentz symmetry could

be broken (even locally) as it happens in theories with more tensor condensates. This may lead to Lorentz violation in proton decay (as first discussed in [11]), a striking possibility since proton decay experiments have assumed so far strict Lorentz invariance, possibly missing already occurring events. On the other hand, the coupling of both θ and ϕ to fermions may introduce such a Lorentz-symmetry breaking also in the matter sector.

Another major issue that we did not mention so far is that a proper understanding of the GGUT breaking mechanism will require a theory of quantum gravity. It is clear that at sufficiently low energy the Planck mass fields decouple and that the remaining ones can be described by an effective field theory. We are assuming that adding the Planck mass fields one can somehow obtain a well defined quantum theory. Asymptotic safety could be of help here, see [21] and references therein. We note finally that if unification works as described here, the mystery of the origin of flavors appears to be even deeper than the issues posed by quantum gravity.

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APPENDIX

A Weyl basis for Euclidean $SO(n)$ gamma matrices can be constructed recursively for even n starting from $n = 2$ with $\gamma_{2,1} = \sigma_1$, $\gamma_{2,2} = \sigma_2$, using the rules

$$\begin{aligned} \gamma_{n,i} &= \gamma_{n-2,i}\,\hat{\gamma}_{n-2} \otimes (-i\sigma_2) \qquad \text{for } i < n-1 \\ \gamma_{n,n-1} &= \mathbf{1}_{d(n-2)} \otimes \sigma_1\,, \\ \gamma_{n,n} &= \hat{\gamma}_{n-2} \otimes \sigma_2\,, \end{aligned} \tag{18}$$

where $d(n) = 2^n$ is the dimension of the representation and $\hat{\gamma}_n = (-i)^{n/2}\Pi_{i=1}^{n}\gamma_{n,i}$ is the chirality matrix. As one checks, it has the right form $\hat{\gamma}_n = \mathbf{1}_{d(n-2)} \otimes \sigma_3$. The generators of the algebra are

$$\Sigma_{n,ij} = \frac{1}{4}[\gamma_{n,i}, \gamma_{n,j}]\,, \tag{19}$$

and are antihermitian and block diagonal.

In signature $(3, 11)$ (3 negative, 11 positive eigenvalues) the gamma matrices are given by

$$\gamma_k = \begin{cases} i\gamma_{14,k} & \text{for } 1 \le k \le 3\,, \\ \gamma_{14,k} & \text{for } 3 < k \le 14 \end{cases} \tag{20}$$

and the definition of $\hat{\gamma}$ has an additional factor i^3 so that $\hat{\gamma} = \Pi_{i=1}^{14}\gamma_i$. The conjugation operations are

$$A = \gamma_1\gamma_3\,, \quad B = \gamma_1\gamma_3\gamma_4\gamma_6\gamma_8\gamma_{10}\gamma_{12}\gamma_{14}\,, \tag{21}$$

and $C = BA^*$.

The explicit gamma matrices are:

$$\begin{aligned}
\gamma_1 &= \;\; i\,\sigma_2 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \\
\gamma_2 &= -i\,\sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \\
\gamma_3 &= -i\,\mathbf{1} \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \\
\gamma_4 &= -\,\sigma_3 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \\
\gamma_5 &= -\,\mathbf{1} \otimes \mathbf{1} \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \\
\gamma_6 &= \;\;\mathbf{1} \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \\
\gamma_7 &= -\,\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \\
\gamma_8 &= \;\;\mathbf{1} \otimes \mathbf{1} \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \\
\gamma_9 &= -\,\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \\
\gamma_{10} &= \;\;\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \\
\gamma_{11} &= -\,\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_2 \otimes \sigma_2 \\
\gamma_{12} &= \;\;\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_2 \\
\gamma_{13} &= \;\;\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_1 \\
\gamma_{14} &= \;\;\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_3 \otimes \sigma_2
\end{aligned} \tag{22}$$

so that one finds:

$$A = -\sigma_3 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \tag{23}$$

$$B = +\sigma_1 \otimes \sigma_1 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbf{1} \tag{24}$$

$$C = -\sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \tag{25}$$

that are real, symmetric (hermitian) and orthogonal.

The Weyl basis described is not unique, and any similarity $\gamma' = S\gamma S^{-1}$ with $[S, \hat{\gamma}] = 0$, preserving the algebra and the Weyl form, trasforms the conjugations as:

$$A' = SAS^\dagger\,, \;\; B' = SBS^{*-1}\,, \;\; C' = SCS^T\,. \tag{26}$$

This freedom has been exploited in the text to adapt the basis to the MW representation and reach the form (1). In particular we used

$$S^{-1} = S_B S_A S_M\,, \tag{27}$$

with

$$\begin{aligned} S_B &= [\sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes (\sigma_1 - i\sigma_2 - \sigma_3 + \mathbf{1}) - \\ &\qquad \mathbf{1} \otimes \mathbf{1} \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes (\sigma_1 + i\sigma_2 - \sigma_3 - \mathbf{1})] \otimes \mathbf{1}\,, \\ S_A &= \mathbf{1}_{64} \otimes (\mathbf{1} - \sigma_3) + F \otimes (1 + \sigma_3)\,, \\ S_M &= \mathbf{1}_{32} \otimes (\mathbf{1}_4 + i\sigma_3 \otimes \mathbf{1})\,. \end{aligned} \tag{28}$$

where $F = i\sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_2$. ($S_B$ and S_A bring B and A to diagonal form, and S_M brings B to the identity.)

A twist was also adopted to reach a standard basis in signature $(3, 1)$.

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